Introduction to Mathematics and Modeling

lecture 7

The method of substitution

UNIVERSITY OF TWENTE.

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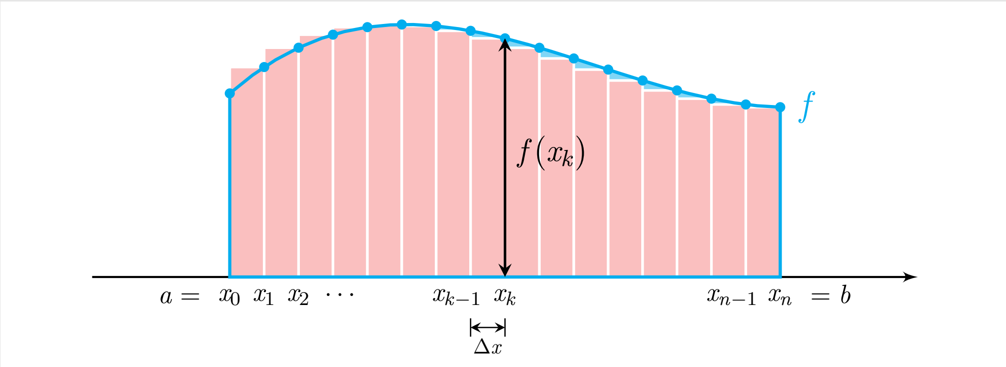
This week

intro

1 Sections 5.1 to 5.4: brief review
2 Section 4.8, 5.5 : indefinite integrals and the substitution method
3 Section 5.6 : definite integrals and the substitution method

Riemann sums



- The **definite integral** of f over $[a, b]$ is

$$\int_a^b f(x)\;\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(x_k\right) \cdot \Delta x$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_k = a + k\Delta x$.

Laws of integration

- Linearity:

$$\int_a^b \alpha\, f(x) + \beta\, g(x)\;\mathrm{d}x = \alpha \int_a^b f(x)\;\mathrm{d}x + \beta \int_a^b g(x)\;\mathrm{d}x$$

- Additivity:

$$\int_a^b f(x)\;\mathrm{d}x = \int_a^c f(x)\;\mathrm{d}x + \int_c^b f(x)\;\mathrm{d}x$$

- Interchanging the upper and lower limit:

$$\int_a^b f(x)\;\mathrm{d}x = -\int_b^a f(x)\;\mathrm{d}x$$

The antiderivative

- The function F is an **antiderivative** for f if

$$F'(x) = f(x)$$

- If F is an antiderivative of f, then also $F(x) + C$ is an antiderivative of f for every constant C.

Integration as the inverse of differentation

The Fundamental Theorem of Calculus

1. *Define the function*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t,$$

then F is an antiderivative of f.

2. *If F is an antiderivative of f then*

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

Notation: $F(b) - F(a) = \Big[\, F(x) \,\Big]_a^b = F(x)\,\Big|_a^b$

The indefinite integral

- The definite integral
$$\int_a^b f(x)\,\mathrm{d}x$$
is a *number*.

Definition

The **indefinite integral of** f is denoted as

$$\int f(x)\,\mathrm{d}x,$$

and is an antiderivative of f plus an arbitrary constant.

- The indefinite integral represents the class of *all* antiderivatives of f.

⚠ The variable x is *not* a dummy variable!

Indefinite integrals of standard functions

$f(x)$	$\int f(x)\,\mathrm{d}x$	
x^α	$\frac{1}{\alpha+1}x^{\alpha+1}+C$	$\alpha\in\mathbb{R}, \alpha\neq -1$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$	
e^x	$e^x + C$	
$\sin(x)$	$-\cos(x)+C$	
$\cos(x)$	$\sin(x)+C$	
$\tan(x)$	$-\ln\lvert\cos(x)\rvert + C$	
$\frac{1}{x^2+1}$	$\arctan(x)+C$	

Antiderivatives of non-standard functions

Methods for computing antiderivatives:

- Educated guess (today)
- Substitution method (this week)
- Integration by parts (next week)
- Trigonometric tricks (next week: §8.2, not in this course: §8.3)
- Partial fraction expansions (not in this course, see §8.4)
- Any combination of the techniques above

Not all antiderivatives can be written in a closed form!

$$\int e^{x^2}\,\mathrm{d}x = ????$$

Integrating by guessing

$$\int e^{3x}\,\mathrm{d}x = ??$$

Integrating by guessing

$$\int \cos\left(\tfrac{1}{2}x - \pi\right)\,\mathrm{d}x = ??$$

Integrating by guessing

$$\int (5x+1)^3\,\mathrm{d}x = ??$$

Always check your answer

Guessing will not always give the desired result:

$$\int (x^2+1)^2 \, \mathrm{d}x = ??$$

- The first guess for the antiderivative is $(x^2+1)^3$.
- Check:

$$\frac{d}{d\,x}(x^2+1)^3 = 3(x^2+1)^2 \cdot 2x = 6x(x^2+1)^2.$$

- The second guess is $\frac{1}{6x}(x^2+1)^3$, but unfortunately

$$\frac{d}{d\,x}\frac{(x^2+1)^3}{6x} = \ldots = \frac{(x^2+1)^2(1-5x^2)}{6x^2},$$

therefore: **ALWAYS CHECK YOUR ANSWER!**

The substitution method

Running the chain rule backwards

Let F be an antiderivative of f, then

$$\int f(g(x))\, g'(x) \, \mathrm{d}x = F(g(x)) + C.$$

Proof

Use the chain rule:

$$\begin{aligned}\frac{d}{d\,x}F(g(x)) &= F'(g(x))g'(x)\\ &= f(g(x))g'(x).\end{aligned}$$

Example 1

$$\int (x^3 + x)^5 (3x^2 + 1) \, \mathrm{d}x = ??$$

Example 2

$$\int \tan x \, \mathrm{d}x = ??$$

The substitution method

$$\boxed{\int f(g(x))\, g'(x)\ \mathrm{d}x = F(g(x)) + C}$$

- Define $u = g(x)$ then

$$\int f(g(x))\, g'(x)\ \mathrm{d}x = F(u) + C$$

- Since $F(x) = \int f(x)\ \mathrm{d}x$ we can also write

$$\boxed{\int f(g(x))\, g'(x)\ \mathrm{d}x = \int f(u) du, \qquad u = g(x)}$$

Differentials

- The notation dx is called a **differential**.
- If $y = f(x)$ then the relation between the differentials dx and dy is given by
$$dy = f'(x)\, dx.$$
- You can memorize this formula by regarding the derivative of f as a fraction:
$$\frac{d\,y}{d\,x} = f'(x) \quad \Longleftrightarrow \quad \mathrm{d}\,y = f'(x)\ \mathrm{d}\,x$$
- Example: let $y = x^2 + x + 1$, then
$$dy = (2x + 1)\, dx.$$

The substitution method

$$\boxed{\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u, \qquad u = g(x)}$$

- Note that $g'(x)dx = du$ and $f(g(x)) = f(u)$.
- Perform the following steps:
  1. Substitute $g(x) = u$.
  2. Get rid of dx by replacing $g'(x)\,dx$ by du, or by replacing dx by $\dfrac{du}{g'(x)}$.
  3. Integrate f with respect to the new variable u.
  4. Replace u by $g(x)$ in the final result.

Example 3

$$\int \cos(3x-1)\,\mathrm{d}x = ??$$

Example 4

$$\int \sqrt{2x+1}\,\mathrm{d}x = ??$$

Example 5

$$\int x\sqrt{2x+1}\,\mathrm{d}x = ??$$

Example 6

$$\int (x^3 + x)^5 (3x^2 + 1)\, \mathrm{d}x = ??$$

See Example 1

Example 7

$$\int x^2 e^{x^3}\, \mathrm{d}x = ??$$

Example 8

$$\int x(x-1)^{10}\,\mathrm{d}x = ??$$

Example 9, alternative 1

$$\int \frac{2x}{\sqrt[3]{x^2+1}}\,\mathrm{d}x = ??$$

The differential

Theorem

Assume that the relation between two variables x and y is defined by the equation

$$h(y) = f(x),$$

then the relation between the differentials of x and y is given by

$$h'(y)\, dy = f'(x)\, dx.$$

- Example: let $x^2 + y^2 = 1$, then $y^2 = 1 - x^2$, hence

$$2y\, dy = -2x\, dx.$$

Example 9, alternative 2

$$\int \frac{2x}{\sqrt[3]{x^2+1}}\,\mathrm{d}x = ??$$

Abuse of the differential

- If $u = g(x)$ then

$$du = g'(x)\,dx.$$

- We abuse the notation by writing

$$d(g(x)) = g'(x)\,dx.$$

Example: if $g(x) = x^2 + 1$, then $d(x^2 + 1) = 2x dx$.

- From right to left: *differentiate*, from left to right: *integrate*:

$d(x^2+1)$	$d(\frac{1}{3}x^3)$	$d(e^{2x})$
$2x\,dx$	$x^2\,dx$	$2e^{2x}\,dx$

- You may add an arbitrary constant to the right hand side:

$$2x\,dx = d\,x^2 = d\,(x^2 + 36).$$

Implicit substitution

Abuse of the differential

If the integrand contains a subexpression $g'(x)$, replace $g'(x)\ \mathrm{d}x$ by the "differential" $d(g(x))$:

$$\boxed{\int f(g(x))\,g'(x)\ \mathrm{d}x = \int f(g(x))\ \mathrm{d}(g(x)) = F(g(x)) + C}$$

where $F' = f$.

- You can replace $g(x)$ by a new variable, say u.

Example 9, alternative 3

$$\int \frac{2x}{\sqrt[3]{x^2+1}}\,\mathrm{d}x = ??$$

Example 10

$$\int \frac{e^x}{e^x+1}\,\mathrm{d}x = ??$$

Example 11

$$\int \frac{1}{x\ln(x)}\,\mathrm{d}x = ?? \qquad (x > 0)$$

Example 12

$$\int \sin x\,\cos x\,\mathrm{d}x = ??$$

We analyze two possible substitutions:

1. Let $u = \cos(x)$ then $du = -\sin(x)dx$.
2. Let $v = \sin(x)$ then $dv = \cos(x)dx$.

Example 12, alternative 1

$$\int \sin x \, \cos x \; \mathrm{d}x = ??$$

Example 12, alternative 2

$$\int \sin x \, \cos x \; \mathrm{d}x = ??$$

Definite integrals

Substitution method for *indefinite integals*:

$$\boxed{\int f(g(x))\, g'(x)\;\mathrm{d}x = F\big(g(x)\big) + C}$$

Substitution method for *definite integals*:

$$\boxed{\int_a^b f(g(x))\, g'(x)\;\mathrm{d}x = F\big(g(x)\big)\,\Big|_a^b = F\big(g(b)\big) - F\big(g(a)\big)}$$

If we write $u = g(x)$ then $F\big(g(b)\big) - F\big(g(a)\big) = F(u)\,\Big|_{g(a)}^{g(b)}$ hence

$$\boxed{\int_a^b f(g(x))\, g'(x)\;\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\;\mathrm{d}u}$$

Example 20

See also example 4.

$$\int_0^4 \sqrt{2x+1}\;\mathrm{d}x = ??$$

Example 21

$$\int_{\frac{1}{2}}^{1} \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,\mathrm{d}x = ??$$